

## Rational Prepayment and the Valuation of Mortgage-Backed Securities

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This Draft: February 15, 1995

### ABSTRACT

This article presents a new model of mortgage prepayments, based on rational decisions by mortgage holders. These mortgage holders face heterogeneous transaction costs, which are explicitly modeled. The model is estimated using a version of Hansen's (1982) generalized method of moments, and shown to capture many of the empirical features of mortgage prepayment. Estimation results indicate that mortgage holders act as though they face transaction costs that far exceed the explicit costs usually incurred on refinancing. They also wait an average of more than a year before refinancing, even when it is optimal to do so. The model fits observed prepayment behavior as well as the recent empirical model of Schwartz and Torous (1989). Implications for pricing mortgage-backed securities are discussed.

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\*This article is based on a chapter from my Ph.D. dissertation. I am grateful for helpful comments and suggestions to Oren Cheyette, Darrell Duffie, David Modest, Chester Spatt (the editor), Robert Whitelaw, Nicky King, and an anonymous referee, as well as seminar participants at U.C. Berkeley, Carnegie-Mellon, Chicago, U.T. Austin, University of Washington, Wharton and Northwestern. I especially wish to thank the members of my dissertation committee, Allan Kleidon, Ken Singleton and Peter Reiss, for their enormous contributions. I thank Goldman, Sachs & Co. for providing the data used in this study, and I gratefully acknowledge financial assistance from the Science and Engineering Research Council, from the American Assembly of Collegiate Schools of Business, from the Berkeley Program in Finance and from the Dean Witter Foundation. Address correspondence to Richard Stanton, Haas School of Business, U.C. Berkeley, 350 Barrows Hall, Berkeley, CA 94720.

A GNMA mortgage-backed security gives its owner a share in the cash flows from a pool of mortgages.<sup>1</sup> To value and hedge these securities requires a model of mortgage prepayment behavior, since this determines the timing of the cash flows. Schwartz and Torous (1989) is a recent example of a large body of literature, both academic and institutional, which empirically models prepayment as a function of some set of (non-model based) explanatory variables. Most such models use either past prepayment rates or some other endogenous variable, such as burnout,<sup>2</sup> to “explain” current prepayment. Their goal is to fit the shape of observed prepayment data, unrestricted by many theoretical considerations. However, since these models are really heuristic reduced form representations for some true underlying process, it is not clear how they would perform in a different economic environment. If the interest rate process were to change, or there were some change to mortgage contract terms, mortgage prepayment behavior would also change. Purely empirical models, including that of Schwartz and Torous, can make no predictions about the magnitude of this change.

Several authors have proposed models of rational mortgage prepayment based on contingent claims pricing theory. In these models, prices and prepayment behavior are determined together, both depending on the assumed interest rate model. Dunn and McConnell (1981a, 1981b) model the optimal prepayment strategy of a mortgage holder who incurs no costs on prepayment, and may face exogenous reasons for prepayment. Their model, however, implies arbitrage bounds on mortgage-backed securities that are often violated in practice. To explain this, Timmis (1985), Dunn and Spatt (1986), and Johnston and Van Drunen (1988) add transaction costs or other frictions that may prevent mortgage holders from exercising their prepayment option and profitably taking advantage of these arbitrage-bound violations. Although these models consistently link valuation and prepayment, their prepayment predictions do not closely match observed prepayment behavior. In their basic forms, with identical transaction costs for all mortgage holders, these models imply that there will either be no prepayment or some “background” level of prepayment until one instant when all

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<sup>1</sup>A mortgage banker or savings and loan issues mortgage loans, either insured by the Federal Housing Administration (FHA) or Farmers Home Administration (FmHA), or guaranteed by the Veterans Administration (VA). It groups the mortgages into a pool, and then obtains mortgage-backed Ginnie Mae certificates from the Government National Mortgage Association (GNMA), which are sold to dealers or investors. GNMA guarantees the payment of interest and principal. The interest rate on a Ginnie Mae security is the interest rate on the mortgages in the underlying pool, less a GNMA guaranty fee of 0.06% per year, and an issuer’s servicing fee of 0.44% per year, a total difference of 0.5%.

<sup>2</sup>Burnout refers to the dependence of expected prepayment rates on cumulative historical prepayment levels. The higher the fraction of the pool that has already prepaid, the less likely are those remaining in the pool to prepay at any interest rate level. See, for example, Richard and Roll (1989).

remaining mortgages in a pool will suddenly prepay. Even with heterogeneous transaction costs,<sup>3</sup> there would still be a single moment for each transaction cost when interest rates hit some critical level, and all mortgage holders with that transaction cost (or lower) would immediately prepay. If interest rates then rose and fell again to this level, there would be no further prepayment observed, since all mortgage holders who would optimally prepay would already have done so. These models do not fully capture all of the empirical features commonly attributed to mortgage prepayment, which include:

1. Seasonality.
2. Some mortgages are prepaid even when their coupon rate is below current mortgage rates.
3. Some mortgages are not prepaid even when their coupon rate is above current mortgage rates.
4. Prepayment appears to be dependent on a burnout factor.

This paper presents a model that is an extension of the rational prepayment models of Dunn and McConnell (1981a, 1981b), Dunn and Spatt (1986), Timmis (1985), and Johnston and Van Drunen (1988). Like these models, and unlike purely empirical prepayment models, it consistently links prepayment and valuation within a single framework, allowing it to address what would happen in the event of a structural shift in the economy. However, it extends these models in several ways. First, it explicitly models and estimates heterogeneity in the transaction costs faced by mortgage holders. Second, mortgage holders make prepayment decisions at discrete intervals, rather than continuously. These two features of the model endogenously produce the burnout dependence noted in previous empirical studies, without the need to specify an ad hoc exogenous burnout factor. They also allow GNMA prices to exceed par by more than the amount of the explicit transaction costs.<sup>4</sup> Finally, the model gives rise to a simple reduced form representation for prepayment. It is estimated using a methodology based on Hansen's (1982) generalized method of moments (GMM), and monthly prepayment data for more than 1,000 mortgage pools over a 6 1/2 year

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<sup>3</sup>See, for example, Archer and Ling (1993).

<sup>4</sup>Dunn and Spatt (1986) note that very high transaction costs are required for their model to explain the observed level of GNMA prices. This is because, in their model, GNMA prices are bounded above by par plus transaction costs. The model described here can exceed that bound because there are implicit costs introduced by mortgage holders' inability to exercise their prepayment option optimally.

period. The estimated level of transaction costs is high at first sight, but seems to be due in large part to the implicit assumption that mortgage holders should be able to borrow at the riskless interest rate. A simple adjustment to take account of the credit risk of mortgage borrowers reduces the estimated cost level substantially. The model produces prepayment behavior that matches closely that actually observed. Measured in terms of percent of variance explained, this model fits the data better than the recent empirical prepayment model of Schwartz and Torous (1989). The estimated transaction costs can be explained as a way of capturing unmodeled credit imperfections of mortgage holders.

The paper is organized as follows. Section 1 lays out the model, first describing the decision process of a single rational mortgage holder. This prepayment decision is used to determine the value of the mortgage holder's liability, and the value of a security backed by a pool of identical mortgages. Heterogeneity in prepayment costs is modeled explicitly, replacing the dependence on endogenous state variables ("burnout") used in previous empirical work. Section 2 describes the detailed implementation of the model. Section 3 deals with the estimation of this model based on its predictions for observed prepayment behavior. Section 4 looks at how to value mortgage-backed securities at their date of issue, and shows that differences in transaction costs have a significant impact on this value. Section 5 presents concluding remarks.

## 1 The Model

### 1.1 Modeling Prepayment

Assume mortgage holders minimize the market value of their mortgage liabilities. They owe the scheduled stream of cash flows on their mortgage, and own a call option which gives them the right to receive an amount equal to each of the remaining mortgage payments, in exchange for payment of the remaining principal plus any applicable transaction costs. Mortgage holder  $i$  has a transaction cost  $X_i$  associated with prepayment. This represents the fraction of the remaining principal balance that the mortgage holder must pay if he or she decides to prepay. While there are monetary costs incurred on refinancing, the cost  $X_i$  also includes the value of non-monetary components reflecting the difficulty and inconvenience of filling out forms, lost productivity etc. In reality, some of these costs will be fixed rather than proportional to the remaining principal balance. Available prepayment data, however,

do not show the size or number of individual mortgages in a GNMA pool.<sup>5</sup> Assuming all costs to be proportional allows us to use homogeneity to predict prepayment without needing to know each mortgage’s principal balance.

Let  $B_t$  denote the value of the underlying bond (the present value of the remaining stream of cash flows on the mortgage) at time  $t$ , and  $F_t$  the remaining principal balance. The mortgage holder has a call option on  $B_t$  with time varying exercise price  $F_t(1 + X_i)$ . The value of the mortgage liability,  $M_t^\ell$ , is

$$M_t^\ell = B_t - V_t^\ell, \tag{1}$$

where  $V_t^\ell$  is the value of the prepayment option to the mortgage holder. Since  $B_t$  does not depend on the mortgage holder’s prepayment decision, minimizing the liability value is equivalent to maximizing the option value.

Besides refinancing for interest rate reasons, the mortgage holder may also prepay for exogenous reasons, such as divorce, job relocation, or sale of the house. The likelihood of exogenous prepayment is described by a hazard function  $\lambda$ . Informally, the probability of prepayment in a time interval of length  $\delta t$ , conditional on not having prepaid prior to  $t$ , is approximately  $\lambda \delta t$ . Hazard functions are discussed in detail in Kalbfleisch and Prentice (1980), and Cox and Oakes (1984). The parameter  $\lambda$  represents a baseline prepayment level, the expected prepayment level when no interest rate driven refinancing should occur.

In previous rational models, mortgage holders reevaluate their prepayment decision constantly. In this model, mortgage holders decide whether to prepay their mortgage at random discrete intervals. This would result, for example, from mortgage holders facing some fixed cost payable when making each decision.<sup>6</sup> Assume the likelihood of making a prepayment decision is governed by a hazard function  $\rho$ . If  $t_i$  is a random decision point, the probability that the next decision is made in a time interval of length  $\delta t$  starting at  $t$  is approximately  $\rho \delta t$ . Figure 1 shows the annualized prepayment rates we would observe if there were no exogenous prepayment and the interest rate remained for 5 years at a level where 50% of the mortgage holders currently in the pool find it optimal to prepay. Different values of  $\rho$  imply different prepayment behavior (and therefore different mortgage values). The smaller the value of  $\rho$ , the smaller the initial prepayment rate, as fewer people prepay each month when

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<sup>5</sup>A richer dataset, with information on loan size, would allow us to test whether the likelihood of prepayment is related to loan size. This is not possible using GNMA data.

<sup>6</sup>This can be regarded as a measure of the difficulty and time involved in deciding whether refinancing is optimal at any time.

it is optimal to do so. A value of 0 corresponds to no prepayment. A value of  $\infty$  implies that everyone prepays instantly, with the prepayment level dropping back to 0 after one month.<sup>7</sup> For values between these extremes, the prepayment rate slowly decays over time to zero.

In principle,  $X_i$ ,  $\lambda$  and  $\rho$  could be functions of other variables, rather than constants. For example, the transaction cost may increase over time as a proportion of the remaining principal balance, if it is partially a fixed sum of money, rather than purely a fraction of the remaining principal; the likelihood of relocation may be related to the level of interest rates, or to business cycle conditions; the time between successive prepayment decisions may be related to the level of interest rates.<sup>8</sup> The implementation of the model is equally simple for any specification, as long as these quantities depend only on interest rates and time. To derive a model that fits the data yet is as parsimonious as possible,  $X_i$ ,  $\lambda$  and  $\rho$  are all assumed to be constant.<sup>9</sup>

Define a prepayment strategy to be a function  $\Omega$  that associates with each possible state  $Y_t$  an element  $\Omega(Y_t, t)$  of the set  $\{0, 1\}$ , where 0 corresponds to no prepayment, and 1 to prepayment.<sup>10</sup> For any given prepayment strategy  $\Omega$ , the prepayment option has a value denoted  $V_t^\ell(\Omega)$  (with dependence on  $Y_t$  suppressed for clarity). Calculation of  $V_t^\ell(\Omega)$ , and therefore determination of the optimal prepayment strategy, requires the specification of a model for interest rate movements and risk preferences. The mortgage holder chooses the optimal prepayment strategy  $\Omega^*$ , defined by

$$V_t^\ell(\Omega^*) \geq V_t^\ell(\Omega), \quad (2)$$

for all  $t$  and  $Y_t$ , and for arbitrary prepayment strategy  $\Omega$ . We write  $V_t^\ell$  for  $V_t^\ell(\Omega^*)$ , and note that the optimal exercise strategy depends only on the transaction cost level and the coupon rate on the mortgage. Since  $V_t^\ell$  is homogeneous of degree 1 in the face

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<sup>7</sup>The models of Dunn and McConnell (1981b), Timmis (1985), Dunn and Spatt (1986), and Johnston and Van Drunen (1988) implicitly set  $\rho = \infty$ .

<sup>8</sup>If interest rates are high, it is relatively unlikely that prepayment will be optimal in the near future, and it makes sense to wait before checking again. If rates are very close to the prepayment boundary, it makes sense to check again soon.

<sup>9</sup>If  $X_i$ ,  $\lambda$  or  $\rho$  varied systematically with other variables, the model's prediction errors when these parameters are assumed constant ought to be correlated with those variables. In separate tests, Stanton (1992) regresses prediction errors from this model against variables including pool size, time since pool issue, housing starts, long and short term interest rates, unemployment, growth in industrial production, and a seasonal dummy variable for summer [as used by Schwartz and Torous (1989)]. None of these variables helps significantly to improve the fit of the model.

<sup>10</sup>We assume no partial prepayments (curtailments) occur. This is justified, except possibly right on the exercise boundary, by the homogeneity of the mortgage holder's problem. If we assume in addition that some of the prepayment costs are fixed, and thus proportionately higher for a curtailment than for a full prepayment, this rules out all curtailments.

value, the value of a mortgage with a face value of \$100 is the same as 100 mortgages with a face value of \$1.

The value of a mortgage-backed security, whose cash flows are determined by the prepayment behavior of the mortgage holder, is  $M_t^a = B_t - V_t^a$  ( $a$  for “asset”). As noted by Dunn and Spatt (1986), there is a difference between the asset and liability values because of the transaction costs associated with prepayment. While these are paid by the mortgagor, and thus increase the value of the liability (reducing the value of the option), they are not received by the investor in the mortgage-backed security. The two values must be calculated simultaneously, since the optimal prepayment strategy of the mortgage holder, determined as part of the liability valuation, in turn determines the cash flows that accrue to the mortgage-backed security.

Each period, given the current interest rate and the transaction cost level of the mortgage holder, the optimal prepayment strategy  $\Omega^*$  determines whether the mortgage holder should refinance. For a given coupon rate, and transaction cost  $X_i$ , there is a critical interest rate  $r_t^*$  such that if  $r_t \leq r_t^*$  the mortgage holder will optimally choose to prepay. Equivalently, for a given coupon rate and interest rate  $r_t$ , there is a critical transaction cost  $X_t^*$  such that if  $X_i \leq X_t^*$  the mortgage holder will optimally prepay. This optimal exercise strategy defines an interest rate dependent hazard function describing the time to prepayment for a single mortgage holder. If it is not optimal for the mortgage holder to refinance, any prepayment is for exogenous reasons, so the hazard rate governing prepayment equals  $\lambda$ . If it is optimal to refinance, the mortgage holder may prepay in the next time interval either for interest rate related or for exogenous reasons. The probability that the mortgage holder does *not* prepay in a (small) time interval of length  $\delta t$  is the probability of neither prepaying for exogenous reasons, nor making an interest rate related prepayment decision during this period,

$$e^{-\lambda \delta t} e^{-\rho \delta t} = e^{-(\lambda + \rho) \delta t}. \quad (3)$$

As  $\delta t$  goes to zero, the probability of prepayment approaches  $(\lambda + \rho) \delta t$ . Thus the hazard rate governing prepayment equals

$$\begin{cases} \lambda & \text{if } r_t > r_t^* \quad (\text{equivalently, } X_i > X_t^*), \\ \lambda + \rho & \text{if } r_t \leq r_t^* \quad (\text{equivalently, } X_i \leq X_t^*). \end{cases} \quad (4)$$

## 1.2 Borrower Heterogeneity and Mortgage Pools

We have so far considered only the prepayment behavior of a single mortgage holder. However, the cash flows that accrue to the owner of a GNMA mortgage-backed security are determined by the prepayment behavior of all mortgage holders in a pool. Valuing a mortgage-backed security backed by a pool of identical mortgage holders, each facing the same transaction costs, and each holding a mortgage with the same coupon rate and the same issue date, is a trivial extension. The liability value per dollar of face value of each mortgage in the pool is the same as that calculated above. The value of a security backed by a pool of mortgages is just a multiple of a single mortgage with face value \$1.

While GNMA regulations insist that the mortgages backing a GNMA mortgage-backed security should be homogeneous, there are inevitably differences between them. If these differences lead to different prepayment speeds, prepayment rates for the pool will exhibit burnout behavior. For example, assuming all pools to be identical, if a large fraction of a pool has already prepaid, those remaining in the pool are likely to be predominantly slow prepayers, and the pool's prepayment rate for any given level of interest rates is likely to be low. Conversely, if only a small fraction of the pool has prepaid, the fraction of fast prepayers left in the pool will be high, and the expected prepayment rate of the pool will be relatively high. To model this behavior, previous empirical papers often assume that a mortgage's prepayment probability depends explicitly on the total prepayment of the pool to date [see, for example, Schwartz and Torous (1989)]. The problem with using such an endogenous state variable is that it is unclear what sort of behavior would lead to the specific burnout representations used.

To treat heterogeneity explicitly, assume that the distribution of prepayment costs among mortgage holders is a beta distribution with parameters  $\alpha$  and  $\beta$ . This distribution is chosen because it has many possible shapes, and constant support.<sup>11</sup> The mean and variance of the beta distribution are

$$\begin{aligned}\mu &= \frac{\alpha}{\alpha + \beta}, \\ \sigma^2 &= \frac{\alpha\beta}{(\alpha + \beta)^2 (\alpha + \beta + 1)}.\end{aligned}$$

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<sup>11</sup>This implies a fixed upper bound on possible transaction cost levels of 100% of the remaining principal balance. However, under the assumptions of the model, a mortgage holder facing a transaction cost close to this level never prepays his or her mortgage for interest rate reasons under reasonable parameter values, so there is no disadvantage to not allowing higher costs.

Given the distribution of transaction costs in a pool, it is now simple to value a GNMA mortgage-backed security backed by that pool. Since the cash flow from the pool is the sum of the cash flows from the individual mortgages, we just value each type of mortgage in the pool, and weight each value by the fraction of the pool of that type.

## 2 Solution of the Model

### 2.1 Interest Rates

To solve the model we must make assumptions about the process governing interest rate movements. We use the Cox, Ingersoll and Ross (1985) one-factor model to characterize nominal interest rate movements. In this model, the instantaneous risk-free interest rate  $r_t$  satisfies the stochastic differential equation

$$dr_t = \kappa(\mu - r_t) dt + \sigma\sqrt{r_t} dz_t. \quad (5)$$

This equation says that, on average, the interest rate  $r$  converges toward the value  $\mu$ . The parameter  $\kappa$  governs the rate of this convergence. The volatility of interest rates is  $\sigma\sqrt{r_t}$ . One further parameter,  $q$ , which summarizes risk preferences of the representative individual, is needed to price interest rate dependent assets.

The parameter values used in this paper are those reported in Pearson and Sun (1989), using data from 1979–1986, which roughly matches the sample period of this study. These values are

$$\begin{aligned} \kappa &= 0.29368, \\ \mu &= 0.07935, \\ \sigma &= 0.11425, \\ q &= -0.12165. \end{aligned}$$

The long run mean interest rate is 7.9%. Ignoring volatility, the time required for the interest rate to drift half way from its current level to the long run mean is  $\ln(1/2)/(-\kappa) \approx 2.4$  years.

Given this model for movements in  $r_t$ , we now need to calculate the value of the mortgage and the optimal exercise strategy for the observed sequence of interest rates. To do this, note that  $V(r_t, t)$ , the value of an interest rate contingent claim paying

coupons or dividends at some rate  $C(r_t, t)$ , satisfies the partial differential equation<sup>12</sup>

$$\frac{1}{2}\sigma^2 r V_{rr} + [\kappa\mu - (\kappa + q)r] V_r + V_t - rV + C = 0. \quad (6)$$

Solving this equation, subject to appropriate boundary conditions, gives the asset value  $V(r_t, t)$ .

## 2.2 Valuation and Optimal Prepayment Strategy

Natural boundaries for the interest rate grid are 0 and  $\infty$ . Rather than solving equation (6) directly, we therefore use the transformation

$$y = \frac{1}{1 + \gamma r}, \quad (7)$$

for some constant  $\gamma > 0$ ,<sup>13</sup> to map the infinite range  $[0, \infty)$  for  $r$  onto the finite range  $[0, 1]$  for  $y$ . The inverse transformation is

$$r = \frac{1 - y}{\gamma y}. \quad (8)$$

Equation (7) says that  $y = 0$  corresponds to “ $r = \infty$ ” and  $y = 1$  to  $r = 0$ . Next, rewrite equation (6) using the substitutions

$$U(y, t) \equiv V(r(y), t), \quad (9)$$

$$V_r = U_y \frac{dy}{dr}, \quad (10)$$

$$V_{rr} = U_y \frac{d^2 y}{dr^2} + U_{yy} \left( \frac{dy}{dr} \right)^2, \quad (11)$$

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<sup>12</sup>We need to assume some technical smoothness and integrability conditions [see, for example, Duffie (1988)].

<sup>13</sup>We shall be using the grid points as approximations for observed interest rates in calculating expected prepayment rates. The finer the grid, the better this approximation. However, the processing time is proportional to each grid dimension. For a given number of grid divisions in the  $y$  direction, the denser the implied  $r$  values are in the range corresponding to observed interest rates (say 4% to 20%), the better will be the approximation of choosing the closest discrete  $r$  value to each observed interest rate. We can affect this density by our choice of the constant  $\gamma$ . The larger the value of  $\gamma$ , the more points on a given  $y$  grid correspond to values of  $r$  less than 20%. Conversely, the smaller the value of  $\gamma$ , the more points on a given  $y$  grid correspond to values of  $r$  greater than 4%. As a compromise between these two objectives,  $\gamma = 12.5$  was used. The middle of the range,  $y = 0.5$ , corresponds to  $r = 8\%$ .

to obtain

$$\frac{1}{2}\gamma^2 y^4 \sigma^2 r(y) U_{yy} + \left(-\gamma y^2 [\kappa \mu - (\kappa + q)r(y)] + \gamma^2 y^3 \sigma^2 r(y)\right) U_y + U_t - r(y)U + C = 0. \quad (12)$$

To value a single mortgage, and simultaneously determine the optimal exercise strategy for that mortgage, we can use a finite difference approximation to solve equation (12). There are several different finite difference approximations. We use the Crank-Nicholson algorithm, which has better stability properties than the simpler explicit finite difference scheme, and a faster order of convergence than the fully implicit method.<sup>14</sup> Using this algorithm involves replacing the derivatives that appear in equation (12) with equations involving the differences between the values of the asset at neighboring points on a discrete grid of  $y$  and  $t$  values. For convenience we use a time interval of one month, yielding a total of 360 intervals in the time dimension.

The Crank-Nicholson algorithm works backward to solve equation (12) one period at a time to calculate the value of the mortgage holder's liability.<sup>15</sup> This gives the value of the mortgage liability conditional on the prepayment option remaining unexercised,  $M_u^\ell(y, t)$ . The value of the mortgage liability if the prepayment option is exercised is the amount repaid, including transaction costs:

$$F(t)(1 + X).$$

It is theoretically optimal to refinance the mortgage if  $M_u^\ell(y, t) > F(t)(1 + X)$ . Otherwise it is optimal not to prepay.<sup>16</sup> The actual value,  $M^\ell(y, t)$ , is a weighted average of  $M_u^\ell(y, t)$  and  $F(t)(1 + X)$ , the weight on  $F(t)(1 + X)$  being the probability that the mortgage is prepaid in month  $t$ . This probability is determined by the parameters  $\lambda$  and  $\rho$ . Let

$$P_e = 1 - e^{-\lambda/12}, \quad (13)$$

the probability of prepayment this month if only exogenous prepayment will occur

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<sup>14</sup>See McCracken and Dorn (1969).

<sup>15</sup>The value of the mortgage in month 360 is 0, since all principal has been repaid. Given known values for the asset or liability at month  $t + 1$ , the algorithm calculates their values at every interest rate level on the grid at month  $t$  by discounting back a weighted average of their possible values at time  $t + 1$ . This is analogous to the "binomial tree" option pricing algorithm. For a detailed discussion of the relationship between binomial methods, discounted expected values, explicit and implicit finite difference methods for the valuation of contingent claims, see Brennan and Schwartz (1978).

<sup>16</sup>To facilitate empirical implementation, we assume that the new contract obtained after refinancing is *not* subject to further refinancing costs.

(i.e. it is not optimal to prepay for interest rate reasons). Let

$$P_r = 1 - e^{-(\lambda+\rho)/12}, \quad (14)$$

the probability of prepayment this month if it is optimal to prepay. The value of the mortgage liability is then

$$M^\ell(y, t) = \begin{cases} (1 - P_e)M_u^\ell(y, t) + P_e [F(t)(1 + X)] & \text{if } M_u^\ell \leq F(t)(1 + X), \\ (1 - P_r)M_u^\ell(y, t) + P_r [F(t)(1 + X)] & \text{otherwise.} \end{cases} \quad (15)$$

To determine the value of an asset (security) backed by this mortgage,  $M^a$ , the process is similar. When the prepayment option is exercised, the security owner receives the remaining principal balance on the mortgage,  $F(t)$ . The value of a security backed by the mortgage is thus

$$M^a(y, t) = \begin{cases} (1 - P_e)M_u^a(y, t) + P_e F(t) & \text{if } M_u^a \leq F(t)(1 + X), \\ (1 - P_r)M_u^a(y, t) + P_r F(t) & \text{otherwise.} \end{cases} \quad (16)$$

This parallels equation (15) above with each  $M^\ell$  replaced by  $M^a$ , but with a different payoff if the mortgage is prepaid. The asset value is less than the value of the mortgage holder's liability at all levels of transaction costs, since the money paid out by the mortgage holder is always less than that received by the owner of the security.

### 2.3 Characteristics of Prepayment Behavior

In this model, if many mortgages have already prepaid, it is likely that the mortgages remaining have relatively high transaction costs. It is therefore likely that in future months relatively low prepayment will occur. The converse is also true. Although there is no explicit burnout factor, the model does exhibit burnout behavior. To give a sense of the different types of prepayment behavior that can be generated by the model, and to see the impact of changing parameter values, Figure 2 illustrates expected monthly prepayment rates for three hypothetical pools of 12.5% mortgages from January 1980 to December 1989. The short term riskless interest rate used is the one month T-Bill return tabulated by Ibbotson Associates. The parameter values  $(\alpha, \beta, \rho, \lambda)$  used for the three plots are A - (0.5, 0.5, 2.0, 0.05); B - (0.5, 0.5, 0.3, 0.05); C - (0.5, 4.0, 0.3, 0.05). Parameters  $\alpha$  and  $\beta$  govern the initial transaction cost distribution, and can be regarded as determining the response at a single time to a change in rates. The parameter  $\rho$  governs how prepayment changes over time.

Ignoring exogenous prepayment, the probability that a mortgage holder prepays in any single month (if it is optimal to do so) is  $1 - e^{-\rho/12}$ . The average time before a given mortgage holder prepays for rational reasons is  $1/\rho$ . For example, for  $\rho = 0.5, 2, 10$ , the probability of prepayment in a single month is 4%, 15% and 57% respectively. The average times before prepayment are 2 years, 6 months and 1.2 months respectively. The expected proportion of the pool prepaying per year for exogenous reasons is  $1 - e^{-\lambda} \approx \lambda$ . All three pools in Figure 2 assume  $\lambda = 0.05$ , so the expected proportion of the pool prepaying for exogenous reasons each year is 4.88%.

The pools labeled A and B have the same distribution of transaction costs, but different  $\rho$  values. Mortgage holders in pool A take an average of 6 months to prepay, while those in pool B take 3 years and 4 months. This implies that pool A should initially (when no prepayment has occurred in either pool) have higher prepayment rates, because a higher proportion of those in the pool who ought to prepay actually do so in any given month (e.g. mid 1980, mid 1982). However, by 1988 the expected prepayment level for pool B is generally higher. This is because almost all of the people in pool A who would choose to prepay have already done so. Also interesting is the fact that prepayment for rational reasons (above the base 4.9% level) occurs at times for pool B when it does not for pool A, such as at the end of 1983. The greater prepayment lag for pool B means they are less able to follow their theoretically optimal option exercise policy (that they would follow if they exercised immediately). This reduces the value of keeping their option unexercised, while the payoff from exercising the option is the same for mortgage holders in both pools. Therefore mortgage holders in pool B may find it optimal to prepay when those in pool A do not.

Pool C has the same  $\rho$  as pool B, but a different initial distribution of transaction costs. The average transaction cost in pool B is 50% of the remaining principal balance, whereas in pool C it is only 11%. Thus, there are always more people who find it optimal to prepay in pool C than in pool B, and the expected prepayment level for the pool is always higher. These comparative dynamics show that one can in principle identify the separate parameters  $\alpha, \beta, \rho, \lambda$  via their implications both for prepayment at a single point in time and for how this changes over time.

### 3 Estimating the Model

Given the hazard functions describing the prepayment behavior of individual mortgage holders, we can in principle write down a likelihood function for a pool's prepayment [see Kalbfleisch and Prentice (1980)]. Parametric heterogeneity can be incorpo-

rated in this likelihood function by integrating over different values of the unobserved parameter(s). Schwartz and Torous (1989) follow this approach, using maximum likelihood to estimate their prepayment model. However, the fact that we do not know the number of mortgages in a GNMA pool leads to a problem. If we assume a single, fixed loan size to determine the number of loans outstanding at any time, the point estimates obtained from maximum likelihood are invariant with respect to this assumed loan size. However, the calculated standard errors are *not* invariant. To avoid this problem, we shall use an alternative approach, based on Hansen's (1982) generalized method of moments (GMM).<sup>17</sup> This provides a means of estimating parameters in a model by matching theoretical moments of the data, as functions of the parameters being estimated, to their sample counterparts. For a full description, see Hansen (1982) or Hansen and Singleton (1982).

### 3.1 Determining the Expected Prepayment Level

We need to identify a set of functions of the parameters and observable data which have an unconditional expectation of zero. Let  $w_{it}$  be the proportion of pool  $i$  prepaying in month  $t$ , where  $i = 1, 2, \dots, N$ . Define  $\Psi_t$  to be the information set at time  $t$ , containing demographic information (date of issue, initial transaction cost distribution, etc.) about each pool and the sequence of observed interest rate values up to and including time  $t$ , and define

$$\boldsymbol{\theta} \equiv (\alpha, \beta, \rho, \lambda).$$

Let

$$\bar{w}_{it}(\boldsymbol{\theta}) = E[w_{it} \mid \Psi_t; \boldsymbol{\theta}], \quad (17)$$

the expected value of  $w_{it}$  conditional on the information set and the parameter values in the prepayment model. If the critical transaction cost level (dependent on the current interest rate) at time  $t$  is  $X_t^*$ , and if  $P_{it}^*$  is the proportion of pool  $i$  with transaction costs less than or equal to  $X_t^*$ , then (suppressing dependence on  $\boldsymbol{\theta}$ )

$$\bar{w}_{it} = P_e(1 - P_{it}^*) + P_r P_{it}^*, \quad (18)$$

where  $P_e$  and  $P_r$  are defined in equations (13) and (14) above. If we calculate the critical cost level for each month, and keep track of the distribution of costs in the pool over time, we can therefore generate a series of expected monthly prepayment

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<sup>17</sup>This also allows us to avoid having to impose arbitrary distributional assumptions.

rates for a pool. However, for an arbitrary continuous transaction cost distribution, searching for the critical cost level and keeping track of the full distribution of costs in the pool over time are numerically very burdensome. To reduce the computation required to a manageable level, we replace the continuous distribution of transaction costs with a discrete approximation.<sup>18</sup> The specific points and probability weights of the discrete distribution should match the underlying continuous distribution as closely as possible under a suitable metric. We define the discrete approximation by a set of  $m$  values  $X_1, X_2, \dots, X_m$ , with associated weights  $c_1, c_2, \dots, c_m$ , giving the estimator

$$\hat{F}(x) = \sum_{j=1}^m c_j I[X_j \leq x], \quad (19)$$

where  $I$  is the indicator function. We choose the specific values of the points and weights to approximate the proportion of mortgages in any range of transaction costs as closely as possible. For a given  $m$ , we choose the  $X_j$  and  $c_j$  to solve

$$\min_{X_j, c_j} \left\{ \sup_{x \in [0,1]} |F(x) - \hat{F}(x)| \right\}, \quad (20)$$

where  $F$  is the true distribution function. The solution to this can easily be shown to be

$$c_j = \frac{1}{m}, \quad (21)$$

$$X_j = F^{-1} \left( \frac{2j-1}{2m} \right), \quad (22)$$

for  $j = 1, 2, \dots, m$ , where

$$F(x) = \int_0^x f(u) du. \quad (23)$$

The  $X_j$  are thus related to the  $m$  quantiles of the distribution. The important feature of this approximation is that the jumps are all of the same size, and are centered around the correct value. This is in contrast to other methods, such as Gaussian quadrature, which provide approximations with other useful features (e.g. closely approximating expected values of polynomials), but have widely varying weights. The approximation used here minimizes the largest possible jump in expected prepayment resulting from a small shift in interest rates.

Assuming the discrete approximation involves  $m$  different transaction cost levels,

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<sup>18</sup>Alternatively, we could think of the true distribution being discrete, though approaching it via a continuous distribution allows us to take advantage of the flexible parametric representation of the beta distribution.

the critical transaction cost level for every time and level of interest rates can be determined by valuing  $m$  mortgages, each on a grid of interest rate and time values, simultaneously determining the optimal exercise strategy for each mortgage. The greater the number of points used, the closer the discrete approximation to the true underlying distribution, but the greater the computational burden. Given the initial distribution of transaction costs defined by the cost levels  $X_j$  and the associated weights  $c_j$ , it is now a simple matter to calculate the expected prepayment level each month from equation (18). Let the proportion of mortgages in pool  $i$  with transaction cost  $X_j$  at time  $t$  be  $c_{jt}$ , ( $j = 1, 2, \dots, m$ ).<sup>19</sup> The proportion of the pool with transaction costs less than or equal to the critical value is

$$P_{it}^* = \sum_{j=1}^m c_{jt} I [X_j \leq X_t^*]. \quad (24)$$

The expected proportion of the pool with transaction cost  $X_j$  at time  $t + 1$  is

$$c_{jt+1} = \begin{cases} \frac{c_{jt}(1-P_r)}{1-\bar{w}_{it}} & \text{if } X_j \leq X_t^*, \\ \frac{c_{jt}(1-P_e)}{1-\bar{w}_{it}} & \text{if } X_j > X_t^*. \end{cases} \quad (25)$$

### 3.2 Moment Restrictions

Write the model's prediction error as

$$e_{it}(\boldsymbol{\theta}) = w_{it} - \bar{w}_{it}(\boldsymbol{\theta}). \quad (26)$$

In matrix form, write

$$\mathbf{E}(\boldsymbol{\theta}) = \begin{pmatrix} e_{00} & e_{01} & \dots & e_{0T} \\ e_{10} & e_{11} & \dots & e_{1T} \\ \vdots & \vdots & \ddots & \vdots \\ e_{N0} & e_{N1} & \dots & e_{NT} \end{pmatrix}, \quad (27)$$

where  $N$  is the total number of mortgage pools and  $T$  the number of months of prepayment data for each pool. We shall estimate the model by choosing parameter values that set the average value of each column of  $\mathbf{E}$  as close to zero as possible. The conditional expectation  $E[e_{it}(\boldsymbol{\theta}_0) | \Psi_t]$  is zero, where  $\boldsymbol{\theta}_0$  is the vector of true values

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<sup>19</sup>So  $c_{j0} = c_j$ .

for the prepayment model's parameters. If  $z_{jt}$  is any element of  $\Psi_t$ , then

$$E [e_{it}z_{jt} \mid \Psi_t] = 0, \quad (28)$$

and so by iterated expectations the unconditional expectation

$$E [e_{it}z_{jt}] = 0. \quad (29)$$

Thus we can in principle form extra moment conditions by using instruments from the information set  $\Psi_t$ , such as past interest rates and prepayment rates. However, since each column of  $\mathbf{E}$  shares a common value of  $t$ , multiplying by any instrument  $z_{jt}$  that varies over time, but is common across all pools (such as an interest rate variable) would result in a constant multiple of the old column, rather an independent moment condition. Using past prepayment rates for each pool would in theory yield additional moment conditions, but even with only a single instrument (the constant 1), there are already 78 moment conditions (one per month of data). As a result, we shall only consider the instrument  $z_{jt} \equiv 1$ .

Define the vector of sample moments  $\bar{e}_N(\boldsymbol{\theta})$  by

$$[\bar{e}_N(\boldsymbol{\theta})]_t = \frac{1}{N} \sum_{i=1}^N e_{it}(\boldsymbol{\theta}), \quad (30)$$

the sample mean of column  $t$  of  $\mathbf{E}$ . Under regularity conditions,  $\bar{e}_N(\boldsymbol{\theta}) \rightarrow E [e_i(\boldsymbol{\theta})]$  almost surely as the number of mortgage pools  $N \rightarrow \infty$ , where  $e_i$  is the  $T$  vector of prediction errors for pool  $i$ . Choosing an estimator  $\hat{\boldsymbol{\theta}}$  to minimize the magnitude of  $\bar{e}_N(\boldsymbol{\theta})$  gives a consistent estimator of  $\boldsymbol{\theta}_0$ . To estimate the parameters of the model, we minimize a quadratic form

$$Q_T(\boldsymbol{\theta}) = \bar{e}_N(\boldsymbol{\theta})' W \bar{e}_N(\boldsymbol{\theta}), \quad (31)$$

where  $W$  is some positive definite weighting matrix. This is done in two stages. First take  $W$  to be the identity matrix, and perform the minimization to derive a first stage estimator of the parameters,  $\hat{\boldsymbol{\theta}}_1$ . Next calculate  $W_N$ , the sample estimator of

$$W_0 = (E [e_i(\boldsymbol{\theta}_0)e_i(\boldsymbol{\theta}_0)'])^{-1}. \quad (32)$$

$W_N$  is given by

$$W_N = \left[ \frac{1}{N} \mathbf{E}(\hat{\boldsymbol{\theta}}_1)' \mathbf{E}(\hat{\boldsymbol{\theta}}_1) \right]^{-1}. \quad (33)$$

Use this as the weighting matrix for the second stage. As long as  $W_N \rightarrow W_0$  almost surely, which will hold if we assume the mortgages to be drawn from a well behaved distribution, the limiting variance-covariance matrix of the GMM estimator is

$$\Sigma_0 = \frac{1}{N} \left[ (E [\partial e_i(\boldsymbol{\theta}_0)' / \partial \boldsymbol{\theta}]) W_0 (E [\partial e_i(\boldsymbol{\theta}_0)' / \partial \boldsymbol{\theta}])' \right]^{-1}, \quad (34)$$

assuming there is no correlation between residuals  $e_{it}$  and  $e_{jt}$  for  $i \neq j$ . This is true under the null hypothesis that the model fully describes the prepayment behavior of all pools, with deviations independent across pools.<sup>20</sup> The sample estimator of this expression gives a consistent estimator for the variance-covariance matrix.

This procedure is different from the usual implementation of time series GMM, where a set of moment conditions is averaged across time. Here we average across pools instead. There are several reasons for doing this. Averaging across time would entail transposing the residual matrix  $\mathbf{E}$  above, and calculating  $N$  different moments (one per pool). The number of mortgage pools exceeds the number of time periods in our sample, so the rank of  $\mathbf{E}$  is at most  $T$ . The calculation of the weighting matrix for the second stage estimator in equation (33) requires that the matrix  $\mathbf{E}'\mathbf{E}$  be invertible. This is true if  $\mathbf{E}$  has full column rank. However, if we were to average across time, the equivalent formula would involve the inverse of  $\mathbf{E}\mathbf{E}'$ , an  $N \times N$  matrix, which can have rank at most  $T < N$ , and is thus not invertible. This problem could potentially be avoided by aggregating data across pools to reduce the number of moments.

Section 2.3 noted that two pools with different prepayment parameters may exhibit prepayment behavior that is not very different on average over time, but changes over time in different ways. Forming the sample moments by averaging the residuals across time throws away all of this variation across time. Both problems are overcome if we transpose the usual residual matrix, and instead average across pools using one moment condition for each month of data.

### 3.3 Data

The prepayment data used for estimation are monthly prepayment rates for a large sample of pools of 12.5% GNMA 30 year single family mortgages. These are newly issued single family residential mortgages, with a coupon rate of 12.5% (corresponding to a coupon rate of 12% on the mortgage-backed security) and an initial term of 30 years. Each mortgage has a default guarantee provided by either the FHA,

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<sup>20</sup>Regressions reported in Stanton (1992) indicate that the sample correlation between errors in separate pools is very small.

VA or FmHA, and the maximum initial mortgage amount is between \$100,000 and \$110,000.<sup>21</sup> The period selected was July 1983–December 1989, to ensure both a long sample period and a large number of pools with prepayment data for the entire period. For each pool, the date of issue, initial balance and coupon rate are known. Also known is the proportion of the initial dollar principal balance remaining in the pool at the end of each month. The number of mortgages in each pool is not known.<sup>22</sup> Pools with missing prepayment data during the sample period were excluded from the study, leaving 1,156 pools in the sample used for estimation.<sup>23</sup> The average annualized prepayment rates between July 1983 and December 1989 varied between 1.6% and 50%. The average proportion of a pool prepaying in any single month was 1.8%, with a standard deviation of 4.3%. The standard deviation of the monthly average prepayment levels (averaged across all 1,156 pools) was 1.6%.

The Cox, Ingersoll and Ross interest rate model describes movements in the short rate. Mortgages are long-term instruments, and their value and optimal strategy are therefore likely to be related to long-term interest rates. In the period subsequent to this study, short rates have become more important due to the steepness of the yield curve and greater use of shorter mortgage instruments such as ARMs. However, this was less significant during the period 1983–89. The Salomon Brothers yield on newly issued 20 year Treasury bonds was used to derive a short rate series to feed into the CIR model. Each month, the short rate  $r$  was calculated which would have produced the observed long bond yield if the CIR model were correct. If the CIR model were strictly correct, this procedure would have no effect, as using any part of the yield curve would produce the same results. However, to the extent that the CIR model does not fully describe movements in the term structure, this procedure allows us to focus on long rate movements.<sup>24</sup> This is especially important because the relationship between long and short rates varied substantially over the period 1983–1989, whereas the CIR model implies that a particular short rate is always accompanied by the same long rate.

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<sup>21</sup>The VA loan limit increased from \$100,000 to \$110,000 in 1981, then again to \$144,000 in 1988. Over the same period, the FHA loan limit was \$90,000, rising to \$101,250 in 1988.

<sup>22</sup>The data for this study were provided by Goldman, Sachs & Co.

<sup>23</sup>Some pools were missing pool factors for certain months. The absence of data was not related to whether the pool had a high or low prepayment rate, so ignoring these pools does not bias our results.

<sup>24</sup>The long interest rate has been used by many other authors, including Schwartz and Torous (1989).

### 3.4 Results

The model was estimated using the 2-stage GMM procedure described above. 78 moment conditions were used, one for each month between July 1983 and December 1989. Table 1 shows the results. The value of  $\rho$  implies that prepayment occurs slowly, even when it is theoretically optimal to refinance. For example, if it is optimal to prepay for a whole year, the probability that a mortgage holder actually prepays during that period is  $1 - e^{-0.6073} = 46\%$ . The average time before a mortgage holder prepays for rational reasons is  $1/\rho = 1$  year, 8 months. This provides evidence against models that assume  $\rho = \infty$ . The estimated value of the parameter  $\lambda$  implies a probability of approximately 3.4% that a mortgage holder prepays in a given year for exogenous reasons, ignoring rational prepayment. This provides evidence against models which assume  $\lambda = 0$ , such as that of Timmis (1985).

The parameters  $\alpha$  and  $\beta$  determine the initial distribution of transaction costs in a mortgage pool. Figure 3 shows the estimated shape of this distribution. Transaction costs are concentrated around the range 30–50%, with a mean value of 41% of the remaining principal balance. This is significantly higher than the explicit monetary costs associated with refinancing, which usually total no more than about 7%. This implies that non-monetary prepayment costs amount to an average of 34% of the remaining principal balance.

In understanding the estimated size of these costs, it is important to note that, besides the explicit and implicit costs already mentioned (the inconvenience of having to go to the bank, fill out forms, take time off work etc.), these transaction costs also serve as a proxy for any other unmodeled or unobservable factors that make prepayment rates lower than the model, as estimated, would predict using more “reasonable” cost levels. There are several such factors. First, the model was estimated assuming mortgage holders to have a 30 year planning horizon. Since the cost of refinancing must be amortized over the remaining life of the mortgage, the shorter the planning horizon, the greater must be the gains to prepayment before these gains offset the costs. Mortgage holders with a shorter planning horizon will thus prepay less often than those with the full 30 year horizon assumed in estimating the model. The first column of Table 2 shows the results of estimating the model using a 10 year investment horizon. The estimates of 0.6452 and 0.0338 for  $\rho$  and  $\lambda$  are similar to those obtained assuming a 30 year horizon. The new estimates for  $\alpha$  and  $\beta$  are 2.4173 and 4.2293, implying an average cost level that has reduced from 41% to 36%.

Credit considerations also serve to reduce prepayment rates. If a mortgagor is not sufficiently credit worthy (or if the underlying house is not sufficiently valuable), he

or she will be unable to refinance, regardless of how low interest rates become.<sup>25</sup> The effective transaction costs for such mortgagors are extremely high, so if they make up a significant fraction of the pool,<sup>26</sup> this will increase the average estimated transaction cost. Another credit related issue is the possibility of mortgagor default. While mortgage-backed securities are insured, so that as far as the investor is concerned, default has the same impact as prepayment, the possibility of default will have some impact on the prepayment behavior of mortgagors. The analysis in this paper implicitly assumes mortgage loans to be (default) risk free. However, mortgagors possess not only an option to prepay their mortgages, but also an option to default on the loan, giving the house to the lender.<sup>27</sup> Ignoring this default option overstates the value of the mortgagor's liability in the absence of prepayment, hence overstating the mortgagor's incentive to prepay, in turn leading to excessive transaction cost estimates. To illustrate the impact of this on the estimated parameter values, the second column of Table 2 shows the results of estimating the model with a 1.5% credit spread, roughly equivalent to that on a low rated corporate bond.<sup>28</sup> Again, the estimates for  $\rho$  and  $\sigma$  are similar to before, but the average estimated transaction cost level is now only 20%.

### 3.4.1 Standard Errors

The standard errors reported in Table 1 appear low.<sup>29</sup> This may be a sample size problem, or may be related to the discrete approximation used in the analysis. The objective function being minimized has discrete jumps, which makes numerical eval-

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<sup>25</sup>A related explanation, believed by many mortgage traders, is that a sizable fraction of mortgage holders never prepay under any circumstances.

<sup>26</sup>Or even just a significant fraction of the borrowers who would tend to prepay slower anyway.

<sup>27</sup>The theoretical interaction between this put option and the prepayment option is considered by Kau, Keenan, Muller and Epperson (1992), assuming a lognormal process for house prices. They find that the value of the default option is generally small for reasonable house price volatilities and loan-to-value ratios, but that for larger volatilities or loan-to-value ratios, its value may approach that of the prepayment option.

<sup>28</sup>The observed long interest rate is increased by this spread. This approximates the rate at which mortgagors could borrow money with no prepayment option, but with the right to default.

<sup>29</sup>These estimates are calculated as if Pearson and Sun's (1989) estimates for the parameters of the CIR interest rate model are constants. Since these are themselves estimates, the standard errors reported in Table 1 are probably understated. Newey (1984) shows in principle how to correct for this. Assuming no correlation between the residuals used to estimate the mortgage model and those used by Pearson and Sun to estimate the CIR model, Newey's equation (8) defines a positive definite correction that needs to be added to the variance matrix calculated above. However, calculation of this correction requires the variance matrix for Pearson and Sun's estimates, which they do not report.

uation of the derivatives used in calculating standard errors rather unreliable.<sup>30</sup> To see whether the discrete approximation is a problem, the estimation was repeated for different numbers of discrete costs and interest rate values. The results are shown in Table 3. The point estimates and standard errors are not identical for different grid sizes, but they do not appear to vary systematically. The reported standard errors do not increase as the grid size increases. Indeed, the reported standard errors for  $\alpha$  and  $\beta$  generally decrease in the number of grid points.

To analyze the reported standard errors further, the prepayment parameters reported in Table 1 were used to generate simulated mortgage prepayment data, which were then used in turn to reestimate the model. Repeating this many times allows us to look at the small sample properties of the distribution of the estimates. Monthly prepayment rates were simulated for 1,000 pools of 1,000 12.5% mortgages between July 1983 and December 1989. This was repeated 100 times, and the model was estimated on each set of simulated data. Table 4 summarizes the results of these estimations. The first column shows the true parameter values used. The second gives the sample mean of the estimated parameter values. The third column gives the sample standard deviation of the estimated parameter values, and the fourth column gives the average reported standard errors, calculated using equation (34). The average parameter estimates are close to the true values, but the average reported standard errors are much lower than the sample values, suggesting that the reported standard errors may be too low. Since this is apparently not due to the grid size, the most likely explanation is that the sample size is not large enough for the asymptotic standard errors to be close to the true values.

The  $\chi^2$  test reported in Table 1 rejects the model. The weighted sum of squares is too large to be due to random fluctuation. One possible explanation is that, since the standard errors seem to be understated, this would probably result in an overstated  $\chi^2$  statistic. To check this, the empirical distribution of the 100 reported  $\chi^2$  values from each estimation performed above using simulated prepayment data was calculated. A  $\chi^2_{74}$  distribution should have a mean of 74. The sample mean was 243. While the theoretical 1% critical level is 105, the sample 1% level was 356. The model is still rejected, but the small sample confidence intervals are very different from their asymptotic counterparts. This rejection may be due to the existence of explanatory variables that are not included in the model.

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<sup>30</sup>Because of this, the Powell algorithm, which does not require the calculation of derivatives, was used to perform all minimizations.

### 3.5 Predictions versus Observed Prepayment Rates

To see how the model's predictions compare with prepayment rates actually observed during the period July 1983–December 1989, Figure 4 shows the average observed annualized monthly prepayment rates for the 1,156 pools used to estimate the model, compared with the average predicted prepayment rates from the model. The  $R^2$  value, calculated from the formula

$$R^2 = 1 - \frac{\text{Variance of prediction error}}{\text{Variance of dependent variable}}, \quad (35)$$

is 91%. For comparison, Figure 4 also shows the fitted values obtained by estimating the Schwartz and Torous (1989) empirical prepayment model using the same data.<sup>31</sup> The  $R^2$  for the Schwartz and Torous model is 88%. Finally, to show the importance of using the long rate, Figure 4 shows the fitted prepayment rates obtained by estimating the rational model using the short rate directly, rather than using an implied short rate derived from the long bond yield. Using the long rate allows the model to match the data substantially better.

## 4 Valuation

The value of a GNMA mortgage-backed security is a weighted sum of the market values of the underlying mortgages. Section 2.2 described how to calculate asset and liability values along with optimal prepayment strategies for individual mortgage holders. To value a security backed by a heterogeneous pool requires calculating the market value of a mortgage with each possible transaction cost level, then weighting each value by the fraction of the pool with that transaction cost level.

Figure 5 shows how the value of a mortgage-backed security is affected by the value of the parameter  $\rho$ , the probability that a prepayment decision gets made per unit time. It plots the market value per \$100 face amount of newly issued 12.5% mortgages against the interest rate  $r$  for different values of  $\rho$ . In each case the transaction cost level is 24%. The value decreases in  $\rho$ , since the lower  $\rho$ , the less the mortgage holder is able to follow the optimal prepayment strategy, the less valuable is the prepayment option, and therefore the more valuable is the mortgage liability.

Looking now at the impact of transaction costs, Figures 6 and 7 plot the market value per \$100 face amount of newly issued 12.5% mortgages against the interest

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<sup>31</sup>Maximum likelihood was used to estimate the model, similarly to Schwartz and Torous (1989).

rate  $r$ . The mortgages in Figure 6 have parameter values  $\rho = \infty, \lambda = 0$ . Those in Figure 7 have  $\rho$  and  $\lambda$  set equal to their estimated values, reported in Table 1. Each graph shows the value for several different transaction cost levels. The mortgage value increases in the level of the transaction costs because a higher transaction cost translates into a higher exercise price for the prepayment option, reducing its value.

The values in Figure 6 exhibit behavior described by Dunn and Spatt (1986). Unlike a plain coupon bond, the GNMA value does not necessarily decrease monotonically in the interest rate  $r$ . This is due to the presence of transaction costs. Consider the case where  $\rho = \infty$ . At very low interest rates, prepayment occurs and the mortgage is therefore worth \$100. At very high interest rates, the value tends to zero. At intermediate rates, the mortgage holder would prepay without transaction costs, but the presence of these costs prevents this prepayment. Restricting the ability of the mortgage holder to exercise his or her prepayment option reduces the value of this option, therefore increasing the market value of the mortgage.

In Figure 7, note that, unlike Figure 6, at low interest rates the security value may now significantly exceed par (\$100) even with zero explicit costs. This occurs because mortgage holders wait more than a year on average before prepaying. This delay precludes them from being able to follow their optimal prepayment strategy, hence reducing the value of their prepayment option, and increasing the value of the mortgage-backed security. Note also that the maximum value (corresponding to infinite transaction costs) has dropped from almost \$150 to about \$140. This is due to the increase in  $\lambda$  from 0 to 0.0345. A non-zero  $\lambda$  has the effect of decreasing the mortgage value when it is trading above par, and increasing the value when it is trading below par, since prepayment always results in a terminal cash flow of \$100.

## 5 Conclusions and Directions for Future Research

This paper presents a new mortgage prepayment model that extends the option-theoretic approach of previous authors. It explicitly models heterogeneity in the costs faced by mortgage holders on prepayment, and assumes that prepayment decisions occur at discrete intervals, rather than continuously, as with previous rational models. These two factors together endogenously produce the burnout dependence noted by previous authors, and allow the model to generate GNMA prices that exceed par even without explicit transaction costs. The model captures many of the empirical regularities observed in prepayment behavior, and possesses a simple moment representation that allows it to be estimated using GMM. The parameter estimates

reported in the paper provide evidence against previous models that assume mortgage prepayment decisions are made continuously. They imply an average time between successive prepayment decisions of more than a year.

This model provides a parsimonious structural means of modeling individual behavior that appears “irrational” according to the predictions of a simple optimal exercise model without transaction costs. Rather than merely curve fitting, allowing the data to predict itself, the rational exercise structure is retained in a modified form. This allows the model to address economic questions that are beyond the scope of purely empirical models. Individuals’ decision processes should remain constant even when the results of those decisions change due to shifts in the economic environment. We could, for example, analyze the impact of a change in the interest rate process, or a change in mortgage contract terms (such as the imposition of an explicit prepayment penalty), on prepayment behavior and security value.

The approach used in this paper has implications for the study of many assets and liabilities with embedded options, whose value is determined by the behavior of a large group of individuals who cannot be counted on to act according to a simple rational model. This includes determining the optimal call policy for corporate bonds, modeling the conversion behavior of the holders of convertible debt, and valuing and hedging investment vehicles such as certificates of deposit (CDs), guaranteed investment contracts (GICs), and single premium deferred annuities (SPDAs).<sup>32</sup>

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<sup>32</sup>With these contracts, an investor makes an initial deposit which grows over time. The investor has the right at any time (possibly subject to some switching cost) to surrender the contract, withdraw the accumulation so far, and reinvest it in a higher yielding vehicle.

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Table 1: Parameter estimates for rational prepayment model

Parameter	Estimate
$\rho$	0.6073 (0.0077)
$\lambda$	0.0345 (0.0007)
$\alpha$	2.9618 (0.0919)
$\beta$	4.2268 (0.1245)
$\chi^2_{74}$ test of overidentifying restrictions	646.5***
Variance of average monthly prepayment rate	.00025
Variance of average prediction error	.00002
$R^2$	0.905

\* Significant at the 10% level.

\*\* Significant at the 5% level.

\*\*\* Significant at the 1% level.

In this table, generalized method of moments (GMM) is used to estimate the parameters  $\rho$  (which measures how likely mortgage holders are to make a prepayment decision in any given period),  $\lambda$  (which measures the probability of prepayment for exogenous (non-interest rate) reasons), and  $\alpha$  and  $\beta$  (which together determine the initial distribution of transaction costs among the mortgage holders in the underlying pools). Standard errors appear in parentheses. Data used for estimation are monthly prepayment rates for 1,156 12.5% GNMA-1 mortgage pools between July 1983 and December 1989.

Table 2: Parameter estimates under alternative assumptions about investment horizon and credit spread

Parameter	Estimate assuming 10 year horizon	Estimate assuming 1.5% credit spread
$\rho$	0.6452 (0.0058)	0.6501 (0.0034)
$\lambda$	0.0338 (0.0004)	0.0374 (0.0006)
$\alpha$	2.4173 (0.0140)	1.0943 (0.0053)
$\beta$	4.2293 (0.0365)	4.2518 (0.0210)
$\chi^2_{74}$	672.4***	735.0***
Var. average monthly prepayment rate	0.00025	0.00025
Var. average prediction error	0.00002	0.00003
$R^2$	0.919	0.894

\* Significant at the 10% level.

\*\* Significant at the 5% level.

\*\*\* Significant at the 1% level.

In this table, generalized method of moments (GMM) is used to estimate the parameters  $\rho$  (which measures how likely mortgage holders are to make a prepayment decision in any given period),  $\lambda$  (which measures the probability of prepayment for exogenous (non-interest rate) reasons), and  $\alpha$  and  $\beta$  (which together determine the initial distribution of transaction costs among the mortgage holders in the underlying pools). Standard errors appear in parentheses. Data used for estimation are monthly prepayment rates for 1,156 12.5% GNMA-1 mortgage pools between July 1983 and December 1989.

Table 3: Parameter estimates for different grid sizes

Parameter	Number of grid points			
	30	100	200	500
$\rho$	0.6073 (0.0077)	0.6556 (0.0077)	0.6321 (0.0093)	0.6851 (0.0096)
$\lambda$	0.0345 (0.0007)	0.0357 (0.0006)	0.0343 (0.0007)	0.0351 (0.0007)
$\alpha$	2.9618 (0.0919)	3.3518 (0.0696)	3.0439 (0.0306)	2.4202 (0.0436)
$\beta$	4.2268 (0.1245)	4.1880 (0.0924)	4.0966 (0.0617)	3.1593 (0.0533)

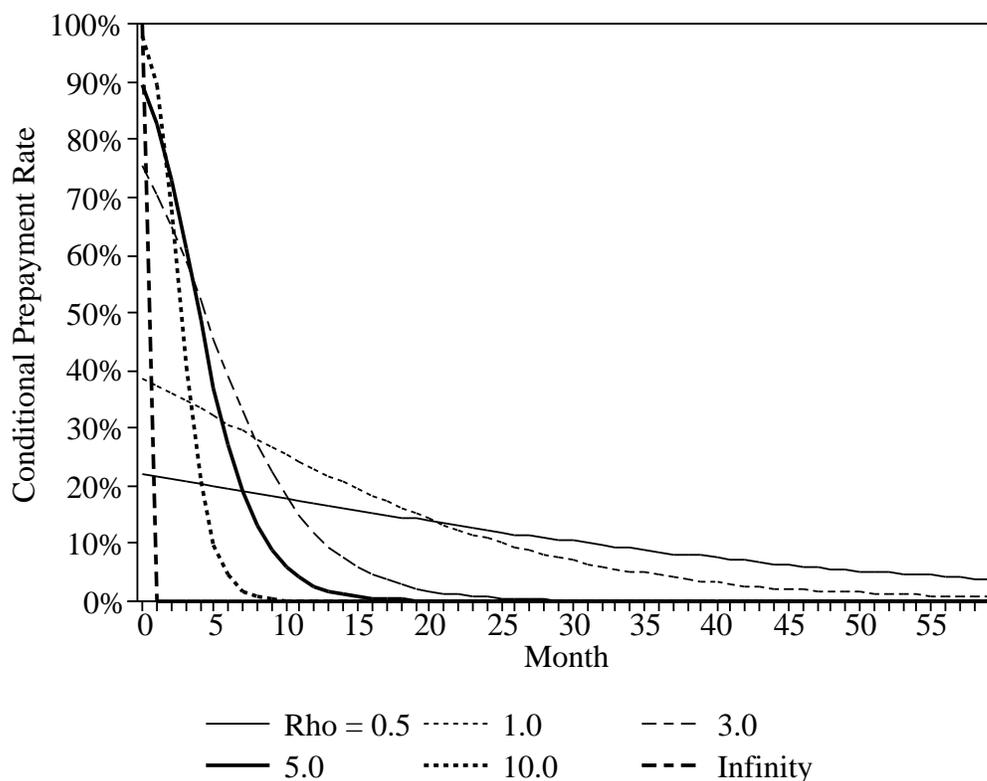
In this table, generalized method of moments (GMM) is used to estimate the parameters  $\rho$  (which measures how likely mortgage holders are to make a prepayment decision in any given period),  $\lambda$  (which measures the probability of prepayment for exogenous (non-interest rate) reasons), and  $\alpha$  and  $\beta$  (which together determine the initial distribution of transaction costs among the mortgage holders in the underlying pools). Standard errors appear in parentheses. Each column shows the results of estimation for a given number of interest rate and transaction cost categories (the same number used for both interest rates and transaction costs). Data used for estimation are monthly prepayment rates for 1,156 12.5% GNMA-1 mortgage pools between July 1983 and December 1989.

Table 4: Simulation results

Parameter	Actual Value	Estimated
$\rho$	0.6073 (0.00251)	0.58076 (0.00060)
$\lambda$	0.0345 (0.00043)	0.03537 (0.00005)
$\alpha$	2.9618 (0.02280)	3.02091 (0.00409)
$\beta$	4.2268 (0.03659)	4.25621 (0.00549)

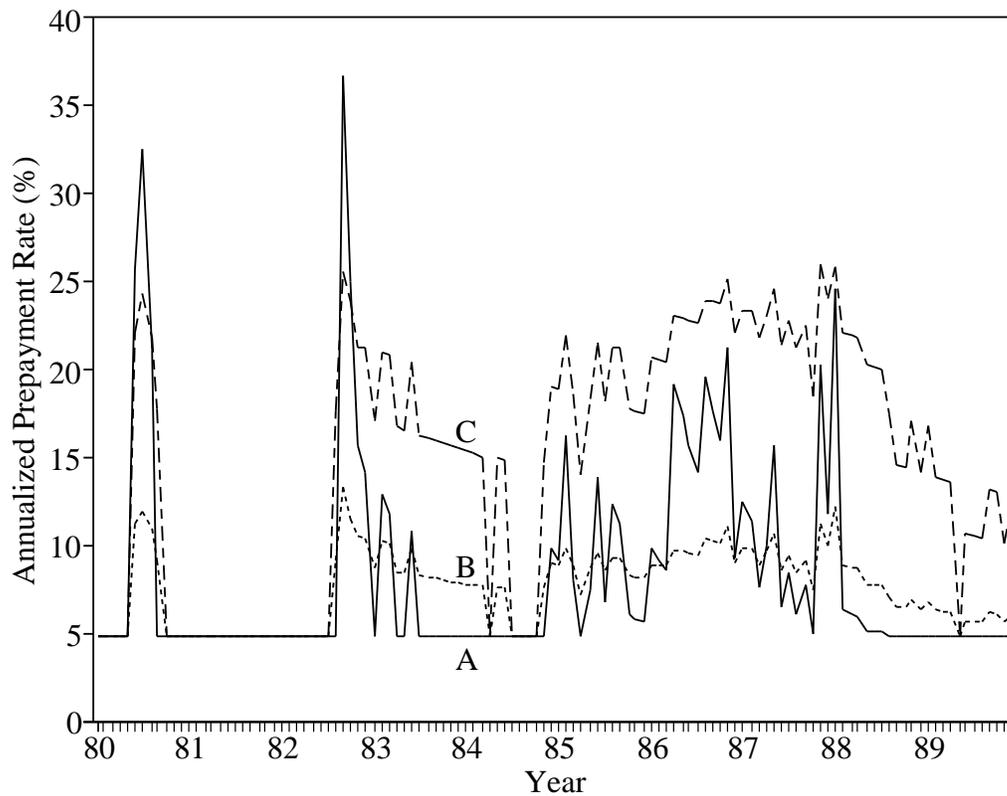
The parameter values in the “Actual” column were used to generate simulated prepayment rates for 1,000 mortgage pools, each containing 1,000 12.5% mortgages. GMM was then used to estimate the model from the simulated data. Parameters estimated are  $\rho$  (which measures how likely mortgage holders are to make a prepayment decision in any given period),  $\lambda$  (which measures the probability of prepayment for exogenous (non-interest rate) reasons), and  $\alpha$  and  $\beta$  (which together determine the initial distribution of transaction costs among the mortgage holders in the underlying pools). The average estimated value (over 100 replications) of each parameter is shown, together with the average reported standard error, and the standard deviation of the estimated values. Data used for estimation are monthly prepayment rates for 1,156 12.5% GNMA-1 mortgage pools between July 1983 and December 1989.

Figure 1: Effect of decision frequency on prepayment behavior.



In this example, 50% of pool initially find it optimal to prepay. Parameter  $\rho$  determines average time between prepayment decisions ( $1/\rho$ ). For large  $\rho$ , every mortgage holder who finds it optimal to prepay does so very quickly. As  $\rho$  approaches zero, the initial prepayment rate is lower, but prepayment continues over a longer period.

Figure 2: Expected annualized prepayment rates for three hypothetical pools of 12.5% mortgages.



Mortgage holders in pool A make more frequent prepayment decisions than those in pools B and C. Those in pool C face lower transaction costs, on average, than mortgage holders in pools A and B. Prepayment rates are calculated conditional on the observed short term interest rate, measured according to the Ibbotson monthly T-Bill return series.

Figure 3: Initial distribution of transaction costs.

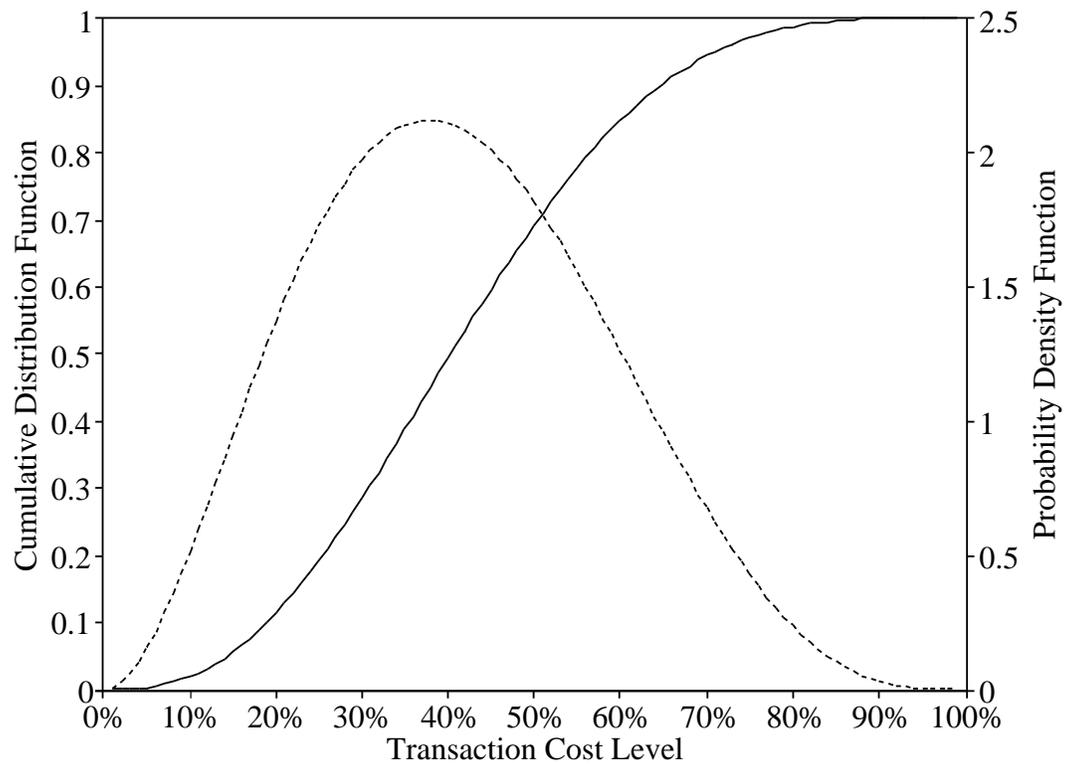
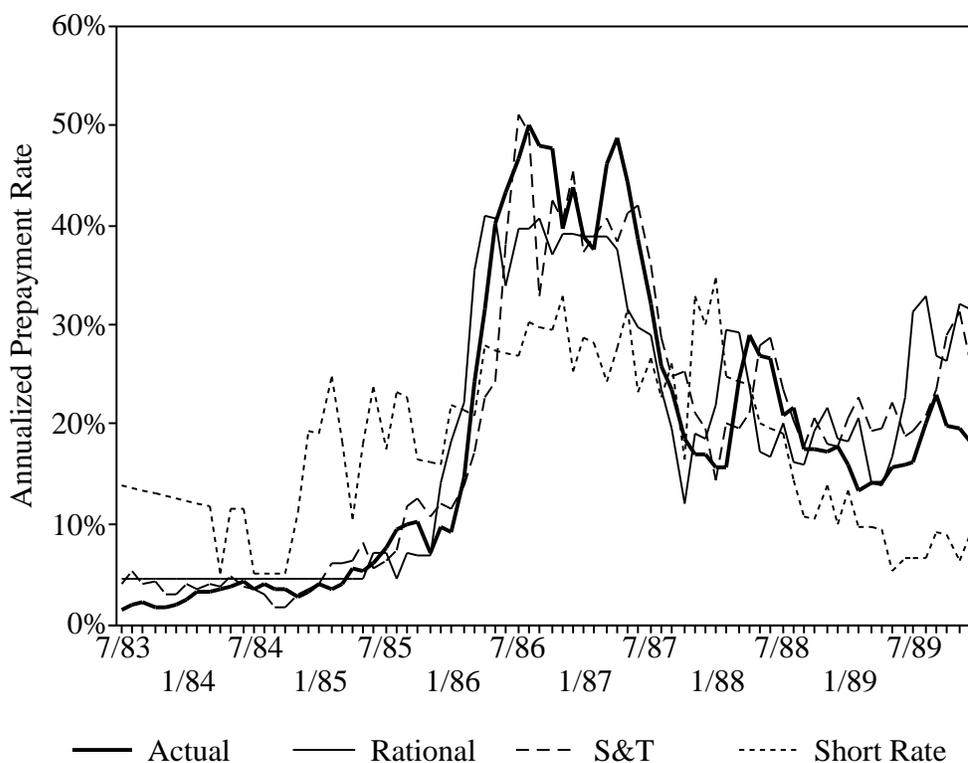


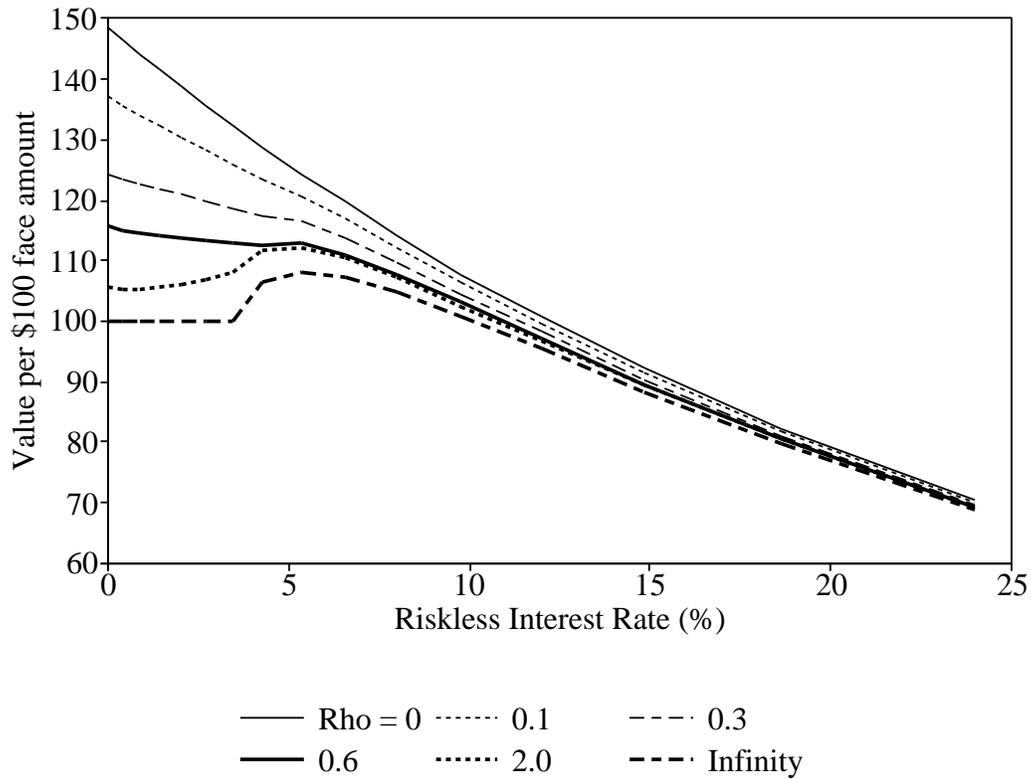
Figure shows distribution based on estimated values of parameters  $\alpha$  and  $\beta$ . Solid line shows cumulative distribution function. Dashed line shows probability density function.

Figure 4: Actual prepayment rates vs. models' predictions.



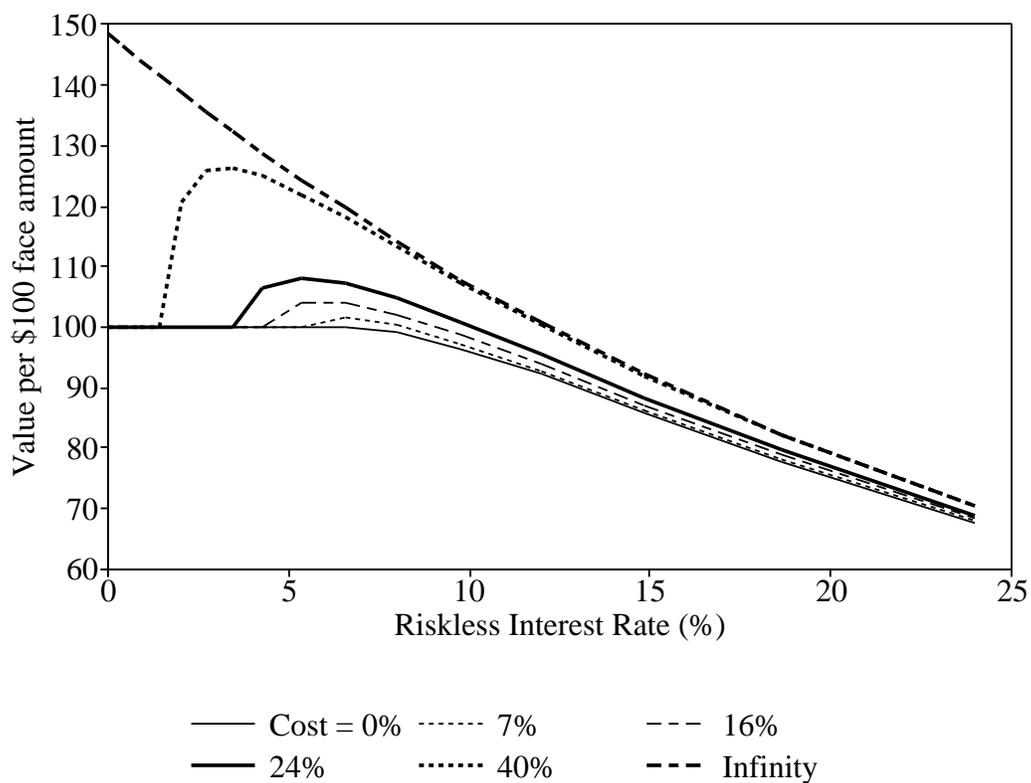
Thick line shows average observed prepayment rates for a sample of 1,156 12% GNMA-1 mortgage pools over the period July 1983–December 1989. Thin solid line shows prepayment rates predicted by rational prepayment model, and dashed line shows prepayment rates predicted by Schwartz and Torous empirical prepayment model. Dotted line shows predictions of rational model, conditional on a short term interest rate rather than a long rate.

Figure 5: Mortgage values for different speeds of prepayment.



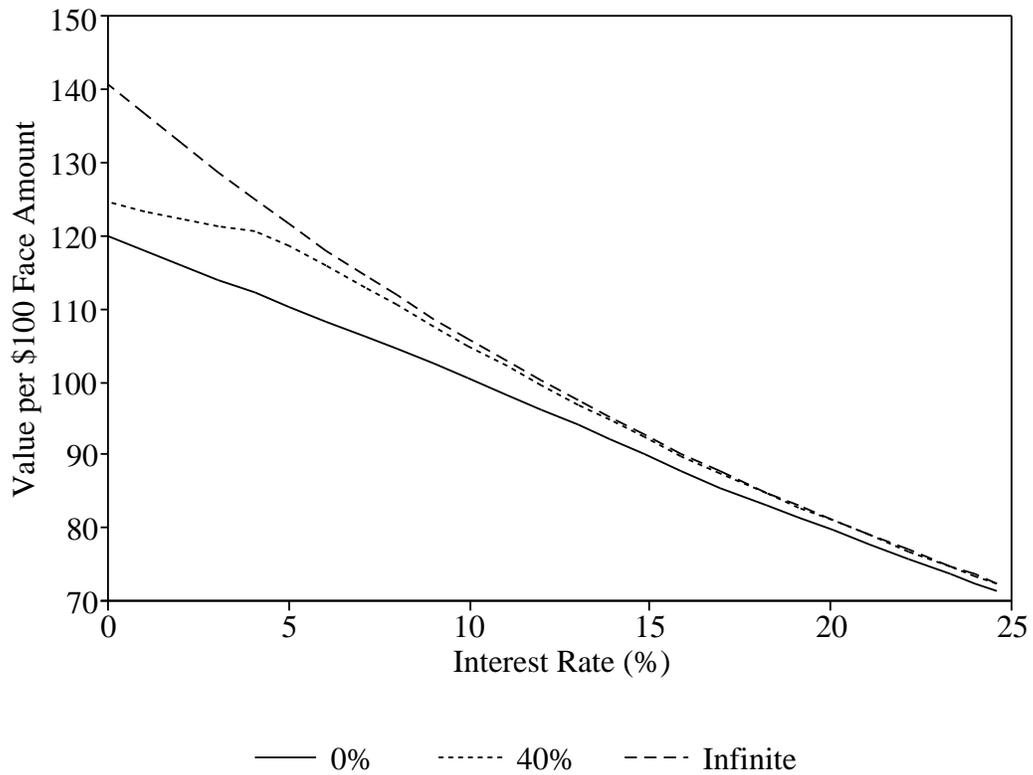
Initial market value per \$100 face amount of 12.5% mortgage for different values of parameter  $\rho$ , which governs average time between successive prepayment decisions. Transaction cost in each case is assumed to be 24% of remaining principal balance.

Figure 6: Mortgage values for different refinancing costs.



Initial market value per \$100 face amount of 12.5% mortgages with different transaction costs payable on prepayment. Mortgage holders are assumed not to prepay for non-interest rate reasons, and to face no delays in exercising their prepayment options.

Figure 7: Mortgage values at estimated parameter values.



Initial market value per \$100 face amount of 12.5% mortgages with different transaction costs payable on prepayment. Likelihood of prepayment for exogenous reasons, and average time between successive prepayment decisions, are assumed equal to their estimated values.